A note on density of cyclic subgroups of a finite group

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Abstract: Let G be a finite group and A be the class of all finite abelian groups. In this paper we show that the image of the function $\alpha : A \to \left[0, \frac{3}{4}\right]$, given by $\alpha(G) = \frac{|C(G)|}{|G|}$ where $C(G)$ denotes the set of cyclic subgroups of G and $G \in A$.

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1. Introduction

Let G be a finite group and $C(G)$ denotes the set of cyclic subgroups of G. The quantity $\alpha(G) = \frac{|C(G)|}{|G|}$ was introduced and studied in [2] by M. Garonzi and I. Lima. We recall the following results from [2].

A. If $(G_i)_{i=\overline{1,k}}$ is a family of finite groups having co prime orders, then

$$\alpha\left(X_{i=1}^{k} G_i\right) = \prod_{i=1}^{k} \alpha(G_i).$$

B. The value $\frac{3}{4}$ is the largest non-trivial accumulation point of the set $\{\alpha(G) | G = finite\ group\}$.
C. It is clear that $0 < \alpha(G) \leq 1$.

We also recall the following Lemma by Marius Tărnăuceanu and Mihai-Silviu Lazorec [1].

Lemma 1.1: Let n be a positive integer, p be an odd prime number and G be a finite p-group of order p^n. Then $\alpha(G) \leq \alpha\left(Z_p^n\right) = \frac{1+\frac{p^n-1}{p-1}}{p^n}$.

Let A be the class of all finite groups. It is obvious that $\alpha(G) \epsilon (0,1]$. Therefore, we consider the function $\alpha : A \to \left[0, \frac{3}{4}\right]$, given by $\alpha(G) = \frac{|C(G)|}{|G|}$ where $C(G)$ denotes the set of cyclic subgroups of G and $G \in A$.

The main objective of this short note is to prove the following theorem.

Theorem1.3. The set $\{\alpha(G) | G = finite\ group\}$ is dense in $\left[0, \frac{3}{4}\right]$. In other words, we prove that the image of α is dense in $\left[0, \frac{3}{4}\right]$.

Firstly, we recall the following results by [1] and [2] respectively.

A. $\alpha\left(Z_p^n\right) = \frac{1+\frac{p^n-1}{p-1}}{p^n}$ where p is a prime and $n \geq 1$ is an integer

B. If G_1 *and* G_2 are two finite groups such that $(|G_1|, |G_2|) = 1$, then $\alpha(G_1 \times G_2) = \alpha(G_1).\alpha(G_2)$.

2. Main Result

In this section, we prove the validity of the theorem1.3 . First of all, we recall the following preliminary results by [11].

Lemma2.1. Let $(x_n)_{n\geq1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} x_n = 0$ and $\sum_{n=1}^{\infty} x_n$ is (convergent) divergent. Then the set containing the sums of all finite subsequences of $(x_n)_{n\geq1}$ is dense in $[0,\infty)$. (A proof is given in [9], see Lemma4.1).

Lemma2.2. Let $(a_n)_{n\geq1}$ and Let $(b_n)_{n\geq1}$ be two sequences of positive real numbers such that $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \beta \,\epsilon\, [0,\infty)$.

If $\beta \,\epsilon\, (0,\infty)$, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same nature.

Proof of the theorem1.3: Consider a sequence $(a_n)_{n\geq1} \subset I_m\alpha$, where $a_n = \alpha\left(X_{i\epsilon I}^k Z_{p_i}^n\right)$, I is finite subset of N^* and p_i is the i^{th} prime number. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\alpha\left(X_{i\epsilon I}^k Z_{p_i}^n\right)$$
$$= \lim_{n\to\infty} \prod_{i\epsilon I} \frac{{p_i}^n + p_i - 2}{{p_i}^n(p_i - 1)}$$
$$= \lim_{n\to\infty} \prod_{i\epsilon I} \frac{1}{p_i - 1}.$$

Hence, we have

$$\left\{\prod_{i\epsilon I} \frac{1}{p_i - 1} \,\middle|\, I \subset N^*, |I| < \infty, p_i = i^{th} prime\ number\right\} \subset \overline{Im\alpha\left(X_{\iota\epsilon}^k\, Z_{p_\iota}^n\right)} = \overline{Im(G)} \subset$$
$$= \left[0, \frac{3}{4}\right].$$

Now, consequently, if we show that the first $\left[0, \frac{3}{4}\right]$, then theorem1.3 holds.

Hence, we prove that

$$\left\{\prod_{i\epsilon I} \frac{1}{p_i - 1} \,\middle|\, I \subset N^*, |I| < \infty, p_i = i^{th} prime\ number\right\} = \left[0, \frac{3}{4}\right].$$

Consider the sequence $(x_i)_{i\geq1} \subset (0,\infty)$ where $x_i = ln\left(\frac{1}{p_i - 1}\right)$.

We have

$$\lim_{i\to\infty} \frac{x_i}{\frac{1}{p_i}} = \lim_{i\to\infty} \frac{ln\left(\frac{1}{p_i - 1}\right)}{\frac{1}{p_i}} = 1.$$

Therefore, since the series $\sum_{i=1}^{\infty}\frac{1}{p_i}$ is convergent by Lemma 2.2 above, we deduce that the series $\sum_{i=1}^{\infty}x_i$ is also convergent. It is obvious that $\lim_{i\to\infty}x_i = 0$, so all the hypotheses of the Lemma2.1 are satisfied. Therefore, we have

$$\overline{\{\sum_{\iota\epsilon I}^{\infty}x_\iota\,|I\subset N^*,|I|<\infty\}} = [0,\infty) \Leftrightarrow \overline{\left\{\sum_{i\epsilon I}^{\infty}ln\left(\frac{1}{p_i-1}\right)\middle|I\subset N^*,|I|<\infty\right\}} = [0,\infty)$$

$$\Leftrightarrow \overline{\left\{\prod_{i\epsilon I}\frac{1}{p_i-1}\middle|I\subset N^*,|I|<\infty,p_i = i^{th}prime\ number\right\}} = [0,\infty).$$

Further, we denote the interval $(0,\infty)$ by Y. Consider the topological spaces (R,T_R) and (Y,T_Y), where T_R is usual topology of R and T_Y is the subspace topology on Y. Note that for a subspace S of R, we have $\overline{S_{T_Y}} = \bar{S}\cap Y$.

Since the function

exp: (R, T_R) → (Y,T_Y), given by $\exp(x) = e^x$, for every real number x, is continuous and $\left[\frac{3}{4},\infty\right)$ is a closed set of R, we have

$$\overline{\left\{\prod_{\iota\epsilon I}\frac{1}{p_\iota-1}\middle|I\subset N^*,|I|<\infty,p_\iota = \iota^{th}prime\ number\right\}} = \left[\frac{3}{4},\infty\right).$$

Note that

$$\overline{\left\{\prod_{\iota\epsilon I}\frac{1}{p_\iota-1}\middle|I\subset N^*,|I|<\infty,p_\iota = \iota^{th}prime\ number\right\}}$$
$$= \overline{\left\{\prod_{\iota\epsilon I}\frac{1}{p_\iota-1}\middle|I\subset N^*,|I|<\infty,p_\iota = \iota^{th}prime\ number\right\}}\cap Y$$
$$= \left[\frac{3}{4},\infty\right)\cap Y$$
$$= \overline{\left[\frac{3}{4},\infty\right)}\cap Y$$
$$= \overline{\left[\frac{3}{4},\infty\right)}_{T_Y}.$$

Hence, if we consider the continuous function $f: Y\to R$, given by $f(y) = \frac{1}{y}$, for every $y\in Y$, we deduce that

$$\overline{\left\{\prod_{\iota\epsilon I}\frac{1}{p_\iota-1}\middle|I\subset N^*,|I|<\infty,p_\iota = \iota^{th}prime\ number\right\}} = \left[0,\frac{3}{4}\right].$$

Consequently, our proof is complete.

 # Further, it is a open problem to study other properties of the function α especially injectivity and surjectivity of α.

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